AMPPZ problems analysis 2016 (Moscow ACM ICPC Workshop edition)

Competition Jury

Institute of Computer Science, University of Wrocław

17 listopada 2016

J – Sharing Chocolate

Problem

We have n rectangular chocolates with sizes $h_i \times w_i$. We perform k *bisplitting* operations: choose the largest chocolate and split it into four roughly equal rectangular parts. Find the largest chocolate left after performing k opeartions for several values of k.

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Suppose we have a single chocolate. Note that after performing several operations the number of different chocolate sizes will be small since each dimension of a new chocolate won't be far from the half the dimension of the old one.

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We will store a `priority_queue` that stores all different chocolate sizes along with the number for each size. Now we can perform the split simultaneously for all chocolates with the same size.
Since there are not too many different sizes, this works fast.

D – Minimal Support of Transportation

For a given weighted graph find the minimal subset of edges that intersects with any MST.

Consider Kruskal algorithm for building MST. On its first step Kruskal's algorithm considers E' — the set of all edges of minimal weight. It then takes the maximal number of edges of E' so that not to form cycles. Then it "merges" the ends of each edge of E' and proceeds to the next weight.

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If E' is not a spanning subgraph, MST will have to contain at least one edge of greater weight. We can skip the edges of E' altogether and proceed to greater weights.

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Fact

The size of a minimal subset intersecting with any spanning tree of a graph is equal to the size of its minimal global cut.

we can find minimal global cut with a number of different algorithms, such as Stoer-Wagner algorithm, Karger's randomized method or simply finding a max-flow between a certain vertex s and all other vertices.

K – Johhny–Bohr model (Maciej Duleba)

Zadanie

Given a small (multi)set of natural numbers B ($|B| \leq 10$), we define a set A as follows:

- $n \in A$ ($n \leq 10^{15}$),
- $\left(\forall_{x \in \mathbb{N} \cup \{0\}}\right) (x \in A) \Rightarrow (\forall_{b \in B}) \left(\frac{x}{b} \in A\right)$.

What is the smallest possible cardinality of A?

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Observation

The answer is always rather small.

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- We can afford to explicitly generate all the elements of A.
- We need to store the already generated **distinct** numbers (for example with a `std::unordered_set`), beware of repetitions in B and the special case of $1 \in B$.

G – Parking Lot (Karol Pokorski)

Problem

Implement a data structure to simulate parallel parking on a line:

- parking in a shortest gap between two cars (if there is a tie: the leftmost),
- leaving the parking.

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- Registration plates are converted to ids (for example, with `std::unordered_map`).
- Gaps are stored in two structures (for example, in a `std::set` with an appropriately modified comparator):
  - sorted according to the positions of their beginning,
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- Gaps are stored in two structures (for example, in a `std::set` with an appropriately modified comparator):
  - sorted according to the positions of their beginning,
  - sorted according to the length (if there is a tie: position of their beginning).
- We also store ids of the currently parked cars.

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- When a car arrives:
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- When a car departs:
    - check if the car is present (and possibly output `BRAK`),
    - remove the adjacent gap from both structures,
    - insert the new gap (or, possible, merge the two adjacent gaps).

C – Generating Polygons (Karol Pokorski)

Problem

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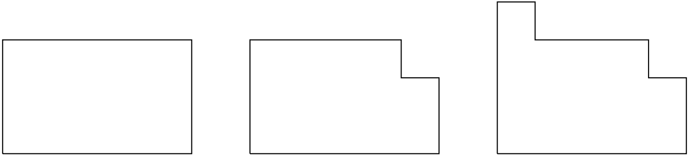
Generate a polygon with all sides parallel to the coordinate axes with a specified area ($1 \leq a \leq 10^{12}$) and perimeter ($4 \leq p \leq 10^6$).

- Multiple possible solutions. We will present just one of them (possibly not the simplest one).
- Find a rectangle with area and perimeter similar to the desired values (with a binary search or simple formula).

We try to make small local changes to the rectangle. The modifications are of two kinds:

- change the area, but leave the perimeter intact,
- change the perimeter by 2 and the area by 1.



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Construct the shortest word with exactly n anagrams ($n \leq 10^{12}$).

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Formula for the number of anagrams

$$m = \binom{a_1}{a_1} \cdot \binom{a_1 + a_2}{a_2} \cdot \binom{a_1 + a_2 + a_3}{a_3} \cdot \ldots$$

where a_i is the number of occurrences of the letter i.

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Spoiler

We will try to backtractk.

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How many distinct letter might be required?
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Answer

No! Word with k distinct letter has at least $k!$ anagrams.

$$15! > 10^{12}$$

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- If $a_1 > 1$ (but not too large) then the situation is also simple because $a_1 \geq a_2 \geq a_3 \geq \dots$ (there are few possibilities for the values a_2, a_3, $\dots$),
- If $a_1 >> 1$ (really large) then the situation is also not too bad because after adding a_2 occurrences of `b` the current number of anagrams $\binom{a_1+a_2}{a_2}$ will be very large. So there are few possibilities for the remaining choices, because the final product of binomial coefficients must be equal to n.

A – Weak pseudorandom generator (Karol Pokorski)

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$$t_n = (a \cdot t_{n-1} + b) \bmod p$$

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- Forget about the modulo for the time being.
- $t_1 = a \cdot t_0 + b$.
- $t_2 = a \cdot t_1 + b = a \cdot (a \cdot t_0 + b) + b = a^2 \cdot t_0 + ab + b$.
- $t_n = a^n \cdot t_0 + b \cdot (a^{n-1} + a^{n-2} + \cdots + a^0)$.

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Solving for a^n

$$a^n = \frac{t_n \cdot (a-1) + b}{t_0 \cdot (a-1) + b}$$

- Now we bring back the modulo and replace division with modular multiplicative inverse.
- n can be extracted by taking a discrete logarithm. This can be solved with the baby–step giant–step algorithm in time $O(\sqrt{p} \cdot \log p)$.
- Beware of division by 0, (multiple) corner cases, and use `long long`.

E – Taxi (Karol Pokorski)

Zadanie

We have n taxi firms ($\leq 10^5$). Each taxi of i-th firm can fit c_i people (≤ 15), and a ride that is x kilometers long is worth $s_i + p_i \cdot (x - 1)$.
We are given q queries ($\leq 10^5$) for driving m_i people ($\leq 10^6$) at d_i kilometers ($\leq 10^6$). For each query find the optimal cost to transporting all people with (possibly several) taxi firms.

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- The cost for hiring each firm is linear in the distance.
- Find the convex hull of these linear functions. We now know the distance intervals such that a particular taxi firm has the optimal price.
- Answer the queries offline, initially sort them by distance.
- We can store pointers to optimal firms for each capacity, and maintain them while increasing the distance. We now only have to consider only ≤ 15 firms with different capacities.

Observation

Each query is now a knapsack problem: we have to collect total mass $\leq m_i$ using items (taxi firms) with weights c_i and costs d_i.

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Problem

Solving this problem naively works in $O(\max(c_i) \cdot m_i)$.

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- Consider a taxi firm that offers the best per-passenger cost. Suppose that this firm has capacity C. Note that we shouldn't use $\geq C$ taxis from any other firm since we can interchange them with the "optimal" firm and optimize the cost.

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Solution

We can now reduce the backpack capacity to C^2. The rest will be filled with taxis of capacity C.
We have to try all possibilities for the total weights of "small" taxis.

F – Ansisocial network (Paweł Gawrychowski)

Zadanie

We have an undirected graph on $n \leq 250\,000$ vertices and $m \leq 2\,000\,000$ edges. We have many operations like: for given $v_1, v_2, \ldots, v_s$ flip edges (v_i, v_j) for all $1 \leq i < j \leq s$. Which vertices are isolated in the resulting graph?

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- To be safe, we can repeat the whole procedure 50 times. In fact, you can perform all these procedures by storing a random `long long` for each vertex and using `xor` instead of modulo 2 sum.

Zadanie

We have a rooted tree on n vertices, $n \leq 5\,000$. Vertex i has bandwidth c_i and initially contains s_i tokens that we want to move to the root. Each second each vertex moves at most c_i tokens it contains to its parent. How many seconds will pass until tokens end up in the root?

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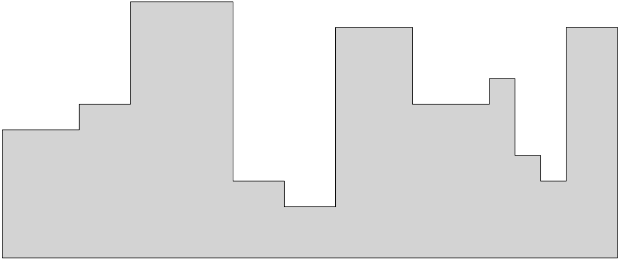

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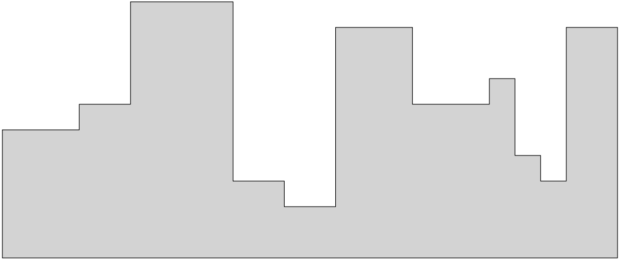
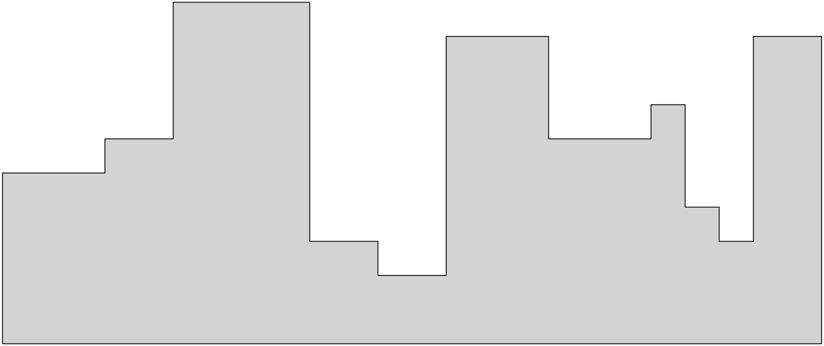


The key observation

This graph contains $O(n)$ steps.

The only operation we need is to build the graph for a vertex u given the graphs for each of its children. First we sum up all the children's functions. Then we take the limit c_u into account.



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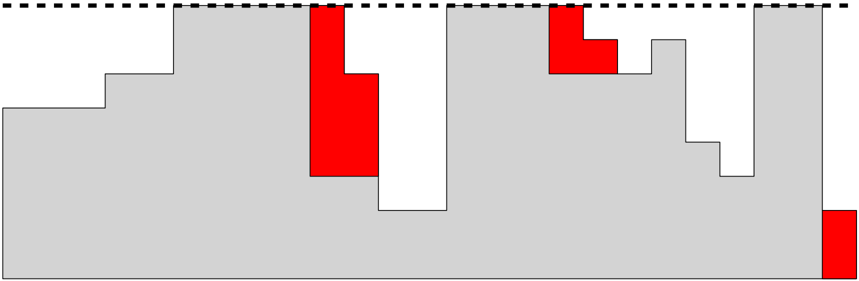

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- Summing two structures is effectively merging two sorted lists and can be done $O(n)$ as well.
- We have to make $O(n)$ operations of these kinds, thus the total complexity is $O(n^2)$ (?).

I – Mountain Hike (Jakub Tarnawski)

Problem

We are given an acyclic graph on n vertices and m weighted edges, $n \leq 1\,000$, $m \leq 10\,000$, with selected vertices s_1, t_1, s_2, t_2. Find two vertex-distjoint paths $s_1 \to t_1$ and $s_2 \to t_2$ with the lowest possible total cost.

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- Sort the vertices by increasing altitude so that the new numbers are $1, 2, \ldots, n$.

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- We have to separately consider the cases when $|\{s_1, t_1, s_2, t_2\}| < 4$.
- We now construct a new graph with vertices being ordere pairs (u, v), $u \neq v$. We want to add edges in such a way that each path from (s_1, s_2) to (u, v) corresponds to to vertex-disjoint paths $s_1 \to u$ and $s_2 \to v$.

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- In a sense, the new graph simulates moving the vertices in parallel.

- There are tow cases. If $u < v$ or $v = t_2$ we try to guess the next vertex on the path that currently ends in u. To do that, we want to try all edges (u, u') such that $u' \neq v$. For each such of the original graph edge we add the edge in the new graph from (u, v) to (u', v) with the same cost.

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- The new graph is acyclic, thus instead of Dijkstra we can implement DP that processed the vertices in the topological order in total linear time.
- New graph has $O(nm)$ edges, thus the solution has the same complexity.

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$n, m \geq 4$

If that condition holds, the answer always exists.

- . . . We can perform brute-force if $n, m \in \{4, 5, 6\}$. But is there a proof? And how to construct the path?

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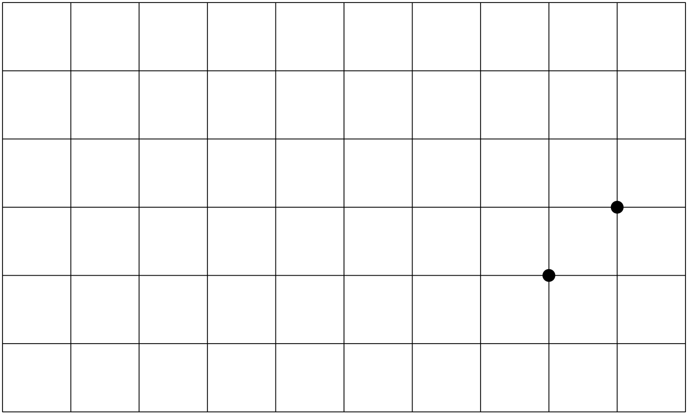
- ... We can perform brute-force if $n, m \in \{4, 5, 6\}$. But is there a proof? And how to construct the path?
- Let's do an induction argument by $n \cdot m$.

Let $n \geq 7$ and the top two rows don't contain start or finish points.



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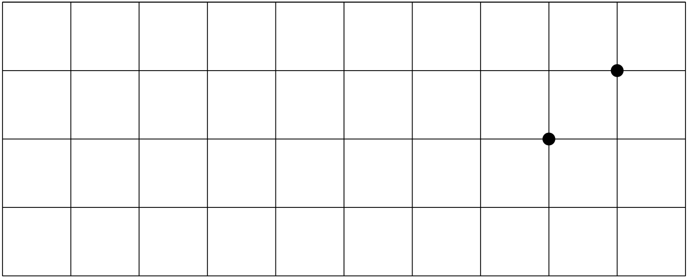

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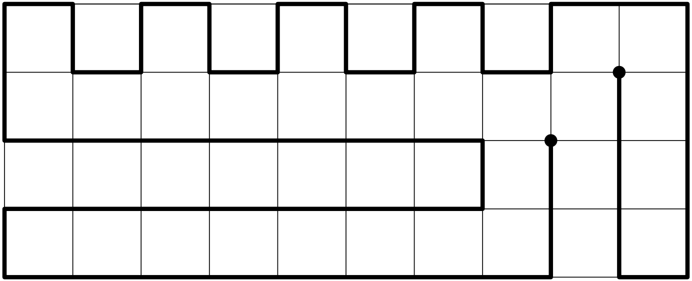

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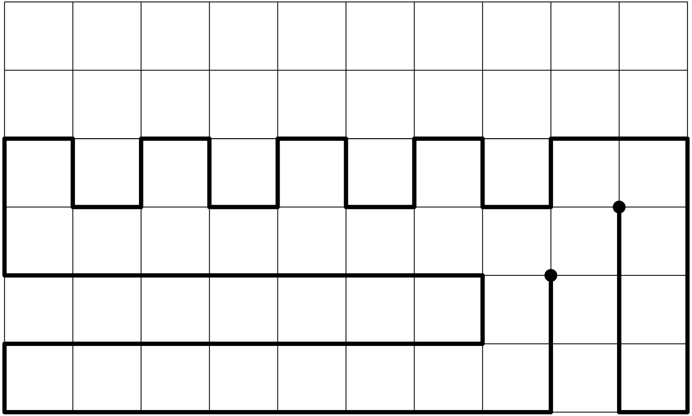

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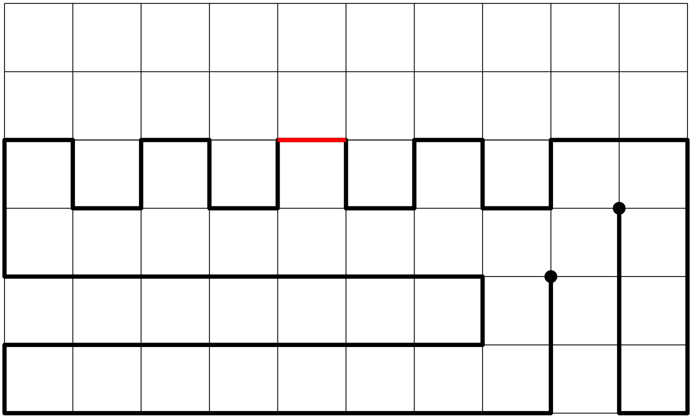

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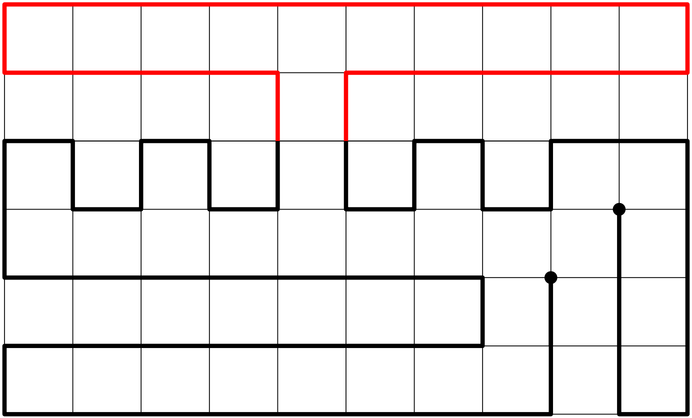

- The same trick also works if the bottom two rows don't contain start or finish.

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- The same trick also works if the bottom two rows don't contain start or finish.
- Thus we only have to deal with the case when the start is in the top two rows, and the finish is not in top three rows (remember that n is not too small).
- We can reduce to the previous case by erasing the top two rows and find the answer recursively on the $(n-2) \times m$ grid, with the start point chosen among the third row in a suitable way.

The same construction applies when $m \geq 7$. If both n and m are small, we can simply perform brute-force. It is true that the answer exists whenever $n, m \geq 4$ and the parity condition is satisfied.

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The resulting solution works in $O(n^2)$.