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No addition operation

Compute the topological order of V. Then $v \rightsquigarrow w$ iff $v < w$.

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Solution — Incremental topological order

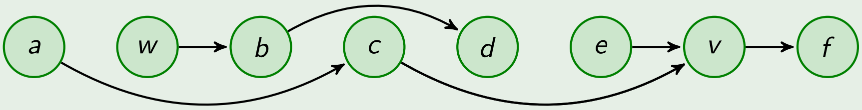
Maintain topological order due to addition of new arcs.

Addition of arc $v \longrightarrow w$

If $v < w$ then do nothing.

Addition of arc $v \longrightarrow w$

The case $w < v$



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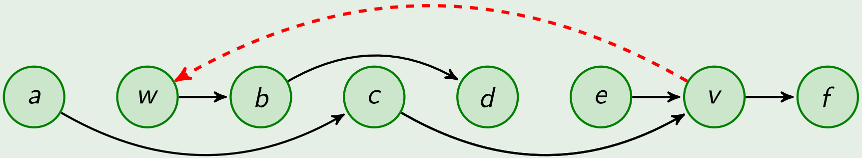

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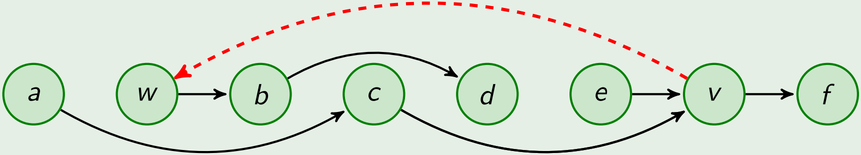

Definitions:

Addition of arc $v \longrightarrow w$

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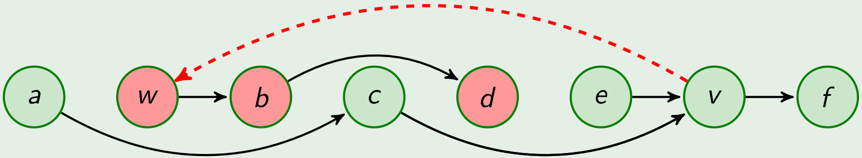

Definitions:

- **Forward vertices** $F = \{w\} \cup \{u\colon w \rightsquigarrow u\}$

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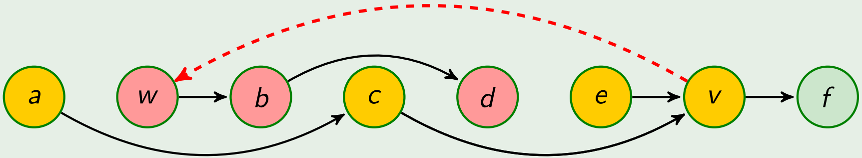

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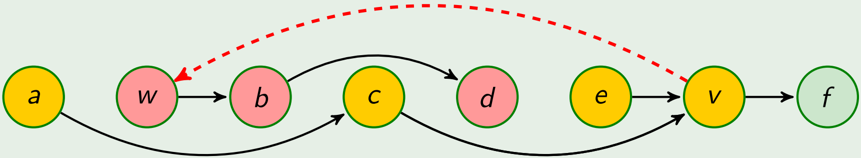

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General Idea

Do bidirectional search forward from w and backward from v until finding either a cycle or a set of vertices whose reordering will restore topological order.

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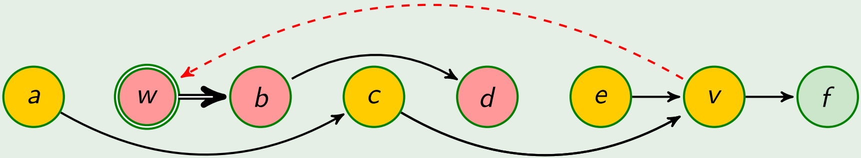
Algorithm A

1. Traverse arcs forward from forward vertices and backward from backward vertices until there is a vertex s such that all forward vertices less than s and all backward vertices greater than s are scanned.

Example



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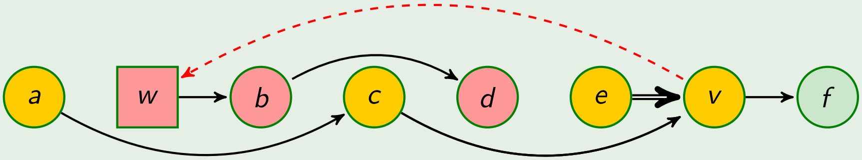

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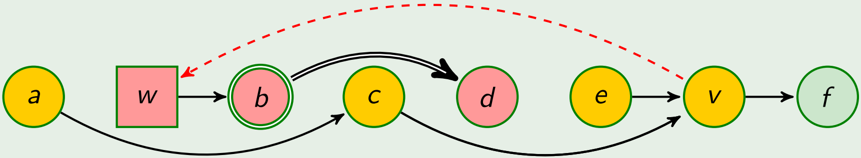

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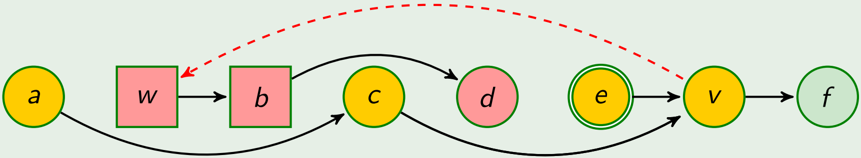

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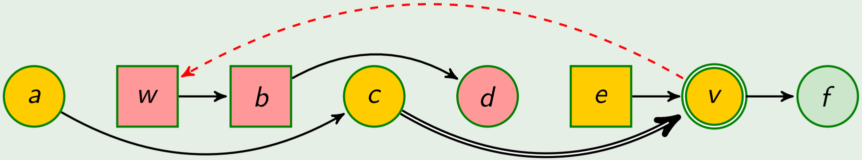

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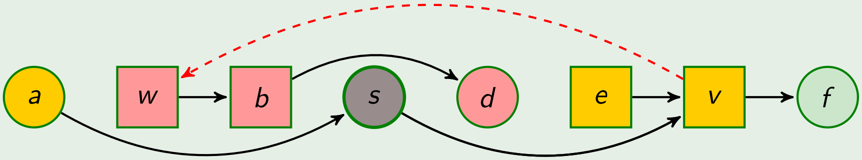

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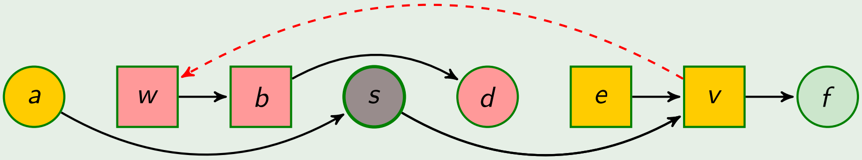
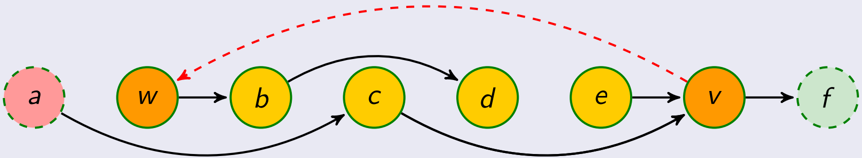
1. Traverse arcs forward from forward vertices and backward from backward vertices until there is a vertex s such that all forward vertices less than s and all backward vertices greater than s are scanned.
2. Let $X = \{x \in F\colon x < s\}$ and $Y = \{y \in B\colon s < y\}$. Find topological orders of O_X and O_Y of the subgraphs induced by X and Y, respectively.
3. Assume s is not forward (the case of s not backward is symmetric). Delete the vertices in $X \cup Y$ from the current vertex order and reinsert them just after s, in order O_Y followed by O_X.

Example



Affected Region

Restrict the search only to the affected region, i.e., the set of vertices between w and v.



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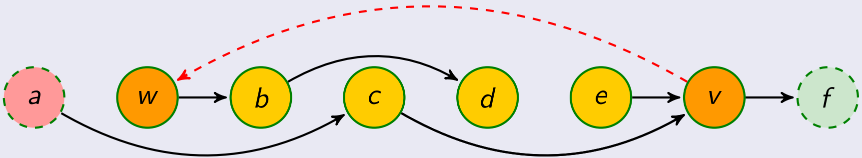

$\Downarrow$

Complexity

$\mathcal{O}(n)$ — amortized time per arc addition

$\mathcal{O}(1)$ — for each query $a < b$

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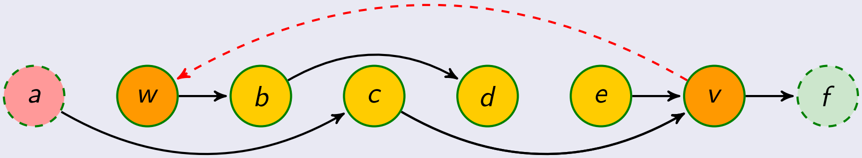

Compatible arcs

We call an arc $u \to x$ traversed forward and an arc $y \to z$ traversed backward compatible if $u < z$.

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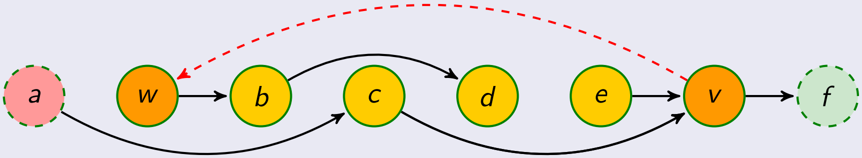

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Lemma

If the searches are compatible, the amortized number of arcs traversed during searches is $\mathcal{O}(m^{1/2})$ per arc addition.

Compatible search

Traverse arcs $u \to x$ forward in non-decreasing order on u and arcs $y \to z$ backward in non-increasing order on z.

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- We maintain the vertex order using a data structure such that testing the predicate $x < y$ for two vertices x and y takes $\mathcal{O}(1)$ time, as does deleting a vertex from the order and reinserting it just before or just after another vertex.