Number Theory Contest Editorial

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A. Sieve It!

Use the linear Eratosthene's sieve to find minimal prime divisors of all numbers. The use the following formulas for $n = p_1^{\alpha_1} \dots p_m^{\alpha_m}$

$s_0(n) = \prod_{i=1}^{m}(\alpha_i + 1)$
$s_1(n) = \prod_{i=1}^{m}\sum_{j=0}^{\alpha_i} p_i^j$
$\varphi(n) = \prod_{i=1}^{m}(p_i - 1)p_i^{\alpha_i - 1}$

B. Cabbages Under Hyperbola

The number of rectangles with rightmost $x = x_r$ is
$x_r(\lfloor n/x_r \rfloor(\lfloor n/x_r \rfloor + 1)/2)$,
so the total number of rectangles is
$\sum_{x=1}^{n} x(\lfloor n/x \rfloor(\lfloor n/x \rfloor + 1)/2)$
This sum be computed in $O(\sqrt{n})$ by grouping summands with $\lfloor n/x \rfloor = k < \sqrt{n}$, and counting all the rest explicitly.

C. Coprime Tuples

By inclusion-exclusion, the answer is $\sum_{d=1}^{n} \mu(d)\lfloor n/d \rfloor^k$. After $\sqrt{n}$-breaking, this appears as

$$\sum_{d=1}^{\sim\sqrt{n}} \mu(d)\left\lfloor\frac{n}{d}\right\rfloor^k + \sum_{x=1}^{\sim\sqrt{n}} x^d\Big(M\Big(\Big\lfloor\frac{n}{x}\Big\rfloor\Big) - M\Big(\Big\lfloor\frac{n}{x+1}\Big\rfloor\Big)\Big)$$

Compute $M(n)$ along with all $M(\lfloor n/k \rfloor)$ in $O(n^{2/3})$, then find the sum in $O(\sqrt{n})$.

D. Count The Semiprimes

The answer is $\sum_{p_j \leqslant \sqrt{n}} \pi(\lfloor n/p_j \rfloor) - j$. Note that the $O(n^{2/3}(\log n)^{1/3})$ method of counting $\pi(n)$ also allows to obtain all $\pi(\lfloor n/x \rfloor)$, given all the needed queries will be preprocessed. Perform preliminary calculation of $dp_{n/p_j,k}$ for all $p_j < \sqrt{n}$ to make sure that queries will be preprocessed. Total complexity does not change.